Tutorial 2

Exercise 1. *For each $n \in \mathbb{N}$, define $S_n = 1 \oplus 2 \oplus \cdots \oplus n$.*

(i) Find the values of S_{33} and S_{34}.

(ii) Consider the n-pile nim game with position $(1, 2, \cdots, n)$.

(1) Find the values of n such that the position $(1, 2, \cdots, n)$ is a P-position.

(2) Find all winning moves from the position $(1, 2, \cdots, n)$ for $n = 33$.

(3) Find all winning moves from the position $(1, 2, \cdots, n)$ for $n = 34$.

Solution: (i) By simple calculation, we have

n	1	2	3	4	5	6	7	8	9	10	11	$12\cdots$
S_n	1	3	0	4	1	7	0	8	1	11	1	$12\cdots$

It is easy to prove by induction that

$$Sn = \begin{cases} n & \text{if } n \equiv 0(\text{mod}4) \\ 1 & \text{if } n \equiv 1(\text{mod}4) \\ n+1 & \text{if } n \equiv 2(\text{mod}4) \\ 0 & \text{if } n \equiv 3(\text{mod}4) \end{cases}.$$

Since $33 \equiv 1(\text{mod}4)$ and $34 \equiv 2(\text{mod}4)$, we have $S_{33} = 1$ and $S_{34} = 35$.

(ii). (1) The set of P-positions is

$$\{(1, 2, \cdots, 4k+3) : k = 0, 1, \cdots\}.$$

(2) Since

$$S_{33} = 1 \oplus 2 \oplus \cdots \oplus 33 = \begin{array}{l} (0,0,0,0,0,1)_2 \\ (0,0,0,0,1,0)_2 \\ \quad\quad\vdots \\ (1,0,0,0,0,0)_2 \\ (1,0,0,0,0,1)_2 \\ \hline (0,0,0,0,0,1)_2 \quad =1 \end{array},$$

we have all winning moves are: $(1,2,\cdots,k-1,k,k+1,\cdots,33) \to (1,2,\cdots,k-1,k-1,k+1,\cdots,33)$ for all odd number k.

(3) Since

$$S_{34} = 1 \oplus 2 \oplus \cdots \oplus 34 = \begin{array}{l} (0,0,0,0,0,1)_2 \\ (0,0,0,0,1,0)_2 \\ \quad\quad\vdots \\ (1,0,0,0,0,0)_2 \\ (1,0,0,0,0,1)_2 \\ (1,0,0,0,1,0)_2 \\ \hline (1,0,0,0,1,1)_2 \quad =35 \end{array},$$

we have all winning moves are: $(1,2,\cdots,32,33,34) \to (1,2,\cdots,3,33,34)$ or $(1,2,\cdots,33,34) \to (1,2,\cdots,32,2,34)$, or $(1,2,\cdots,33,34) \to (1,2,\cdots,33,1)$.

Sprague-Grundy function.

Definition 1. *Let X be the set of all possible positions of a combinatorial game. The S-G function is a map $g : X \to \mathbb{N}$ defined by*

(i) $g(x) := 0$ if x is a terminal position.

(ii) $g(y) = \min\{k \geq 0, k \notin \{g(x) : x \text{ is a follower of } y\}\}$.

The S-G function is a useful tool designed to find the P-positions of a game, as we have the following proposition.

Proposition 2. *Let $\mathcal{P}$ denote the set of P-positions and let g be the S-G function of a game. Then we have*

$$\mathcal{P} = \{x \in X : g(x) = 0\}.$$

Exercise 2. *Consider the subtraction game with $S = \{1, 3, 6\}$.*

(i) Find $g(6)$, $g(13)$ and $g(50)$.

(ii) Find all winning moves from the position that there are 50 chips.

(iii) Find the set of P-positions and give a proof.

Solution (i) Note that the only terminal position is 0. By backwards induction, we have

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	$\cdots$
$g(x)$	0	1	0	1	0	1	2	3	2	0	1	0	1	0	1	2	3	2	$\cdots$

It is easy to see that g is periodic with period 9. Indeed,

$$g(x) = \begin{cases} 0 & \text{if } x \equiv 0, 2 \text{ or } 4 (\text{mod} 9) \\ 1 & \text{if } x \equiv 1, 3 \text{ or } 5 (\text{mod} 9) \\ 2 & \text{if } x \equiv 6 \text{ or } 8 (\text{mod} 9) \\ 3 & \text{if } x \equiv 7 (\text{mod} 9) \end{cases}$$

Hence we have $g(6) = 2$, $g(13) = 0$ and $g(50) = 1$.

(ii) All winning moves from position 50 are removing 1 or 3 chips.

(iii) We claim that the set of P-positions is given by

$$\mathcal{P} = \{k \in \mathbb{N} : k \equiv 0, 2 \text{ or } 4(\text{mod} 9)\}.$$

Proof of the claim: (1). The only terminal position $k = 0$ is in $\mathcal{P}$. (2). If $k \equiv 0, 2$ or $4(\text{mod} 9)$, then we have $k - 1 \equiv 8, 1$ or $3(\text{mod} 9)$, $k - 3 \equiv 6, 8 \text{or} 1(\text{mod} 9)$ and $k - 6 \equiv 3, 5$ or $7(\text{mod} 9)$. Hence any position in $\mathcal{P}$ can only be removed to a position outside $\mathcal{P}$. (3). If $k \not\equiv 2(\text{mod} 9)$, $k \not\equiv 4(\text{mod } 9)$ and $k \not\equiv 6(\text{mod} 9)$, it is easy to see that at least one of $k - 1$, $k - 3$ and $k - 6$ is in $\mathcal{P}$. By the characterization of P-positions, we finish the proof.